

# EQUIVARIANT CHARACTERISTIC CLASSES IN THE CARTAN MODEL

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*Dedicated to the memory of V. K. Patodi*

ABSTRACT. This note shows the compatibility of the differential geometric and the topological formulations of equivariant characteristic classes for a compact connected Lie group action.

Suppose  $G$  and  $S$  are two compact Lie groups, and  $P$  and  $M$  are manifolds. A principal  $G$ -bundle  $\pi : P \rightarrow M$  is said to be  $S$ -equivariant if  $S$  acts on the left on both  $P$  and  $M$  in such a way that

a) the projection map  $\pi$  is  $S$ -equivariant:

$$\pi(s.p) = s.\pi(p) \quad \text{for all } s \in S \text{ and } p \in P;$$

b) the left action of  $S$  commutes with the right action of  $G$ :

$$s.(p.g) = (s.p).g \quad \text{for all } s \in S, p \in P, \text{ and } g \in G.$$

An  $S$ -equivariant principal  $G$ -bundle  $\pi : P \rightarrow M$  induces a principal  $G$ -bundle  $\pi_S : P_S \rightarrow M_S$  over the homotopy quotient  $M_S$ . In the topological category, the *equivariant characteristic classes* of  $P \rightarrow M$  are defined to be the corresponding ordinary characteristic classes of  $P_S \rightarrow M_S$ . Thus, the equivariant characteristic classes are elements of the equivariant cohomology ring  $H_S^*(M)$ .

There is also a differential geometric definition of equivariant characteristic classes in terms of the curvature of a connection on  $P$  ([3])([4]). However, there does not seem to be an explanation or justification in the literature bridging the two approaches. The modest purpose of this note is to show the compatibility of the usual differential geometric formulation of equivariant characteristic classes with the topological one. We have also tried to be as self-contained as possible, which partly explains the length of this article.

Let us first recall the situation for ordinary characteristic classes. Here the famous Chern-Weil construction represents the ordinary characteristic classes of a principal  $G$ -bundle  $\pi : P \rightarrow M$  by differential forms as follows. Fix a connection  $\Theta$  on  $P$ . Then its curvature  $K$  is a 2-form on  $P$  with values in the Lie algebra  $\mathfrak{g}$  of  $G$ . For each  $\text{Ad } G$ -invariant polynomial  $f$  on  $\mathfrak{g}$ , one shows that  $f(K)$  is a basic form on  $P$  and so is the pullback  $\pi^*\Lambda_f$  of a form  $\Lambda_f$  on  $M$ . Moreover,  $\Lambda_f$  is closed and the cohomology class of  $\Lambda_f$  in  $H^*(M)$  is independent of the connection.

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*Date:* January 31, 2001.

*Key words and phrases.* Equivariant cohomology, equivariant characteristic classes, Cartan model, Weil model.

*2000 Mathematics Subject Classification.* Primary: 57R20; Secondary: 55R40, 55N25.

This work and the first author were supported in part by the National Science Foundation. The second author was supported in part by the Institut Henri Poincaré, the Ecole Normale Supérieure, and the Ministère de l'Enseignement Supérieur et de la Recherche, France.

The characteristic class  $c_f(P)$  of  $P$  associated to  $f$  is the cohomology class  $[\Lambda_f] \in H^*(M)$ .

For an  $S$ -equivariant principal  $G$ -bundle  $P \rightarrow M$ , the equivariant characteristic classes live in the equivariant cohomology ring  $H_S^*(M)$ . Since the equivariant cohomology of  $M$  is the cohomology of its Cartan model, it is natural to ask if equivariant characteristic classes can be constructed in the Cartan model out of the curvature of a connection on  $P$ .

This is indeed possible. Assume that the Lie group  $S$  is compact connected with Lie algebra  $\mathfrak{s}$ . Then there exists an  $S$ -invariant connection  $\Theta$  on the  $S$ -equivariant principal  $G$ -bundle  $\pi : P \rightarrow M$ . Let  $K$  be the curvature form of the  $S$ -invariant connection  $\Theta$ . For each  $X \in \mathfrak{s}$ , define  $L_X : P \rightarrow \mathfrak{g}$  by  $L_X = -\iota_X \Theta$ . We will show that the equivariant characteristic class associated to an  $\text{Ad}(G)$ -invariant polynomial  $f$  on  $\mathfrak{g}$  is represented by  $f(K + L_X)$  in the Cartan model. In terms of a basis  $X_1, \dots, X_\ell$  for  $\mathfrak{s}$  and dual basis  $u_1, \dots, u_\ell$  for  $\mathfrak{s}^*$ , this element is  $f(K + \sum u_i L_{X_i})$ .

The outline of the proof is as follows. Since the total space  $ES$  of the universal  $S$ -bundle is not a manifold, the first obstacle in a differential geometric treatment of equivariant characteristic classes is that neither  $P_S$  nor  $M_S$  are manifolds. Nonetheless, by approximating  $ES$  by finite-dimensional manifolds  $ES(r)$ ,  $r = 1, 2, \dots$ , we can approximate the homotopy quotients  $P_S$  and  $M_S$  by the manifolds

$$P^r := ES(r) \times_S P, \quad M^r := ES(r) \times_S M.$$

The natural map

$$\begin{aligned} \tau : P^r &\rightarrow M^r \\ [x, p] &\mapsto [x, \pi(p)], \quad x \in ES(r), p \in P \end{aligned}$$

is again a principal  $G$ -bundle and is a finite-dimensional approximation to  $\pi_S : P_S \rightarrow M_S$ .

On the principal  $G$ -bundle  $ES(r) \times P \rightarrow ES(r) \times M$  we are able to write down a connection which is basic with respect to the  $S$ -action. This connection then descends to a connection  $\Xi_r$  on  $P^r \rightarrow M^r$ . By computing explicitly the curvature  $K_r$  of the connection  $\Xi_r$ , it is easy to determine the element of the Weil model of  $H_S^*(M)$  that restricts to the characteristic classes  $[f(K_r)] \in H^*(M^r)$  for all  $r$ . Using the Mathai-Quillen isomorphism between the Weil model and the Cartan model ([10]), we can then write down the element of the Cartan model that represents the equivariant characteristic class  $c_f(P)$ .

**Notations.** Since there are two groups acting on  $P$ , we need to be careful to distinguish between them. A general element of  $S$  will be denoted  $s$ , and a general element of  $G$  will be denoted  $g$ . Let  $\ell_s$  and  $r_g$  denote left multiplication by  $s$  and right multiplication by  $g$ , respectively. The dimension of  $S$  will be  $\ell$  and the dimension of  $G$  will be  $n$ . A basis for the Lie algebra  $\mathfrak{s}$  of  $S$  is  $X_1, \dots, X_\ell$ , and a basis for the Lie algebra  $\mathfrak{g}$  of  $G$  is  $e_1, \dots, e_n$ . We denote by  $\tilde{Y} = \tilde{Y}_P$  the fundamental vector field on  $P$  corresponding to  $Y \in \mathfrak{s}$  under the  $S$ -action, and by  $Z^* = Z_P^*$  the fundamental vector field on  $P$  corresponding to  $Z \in \mathfrak{g}$  under the  $G$ -action.

## 1. EQUIVARIANT COHOMOLOGY

We begin with a rapid review of the basic constructions of equivariant cohomology.

Suppose  $M$  is a CW-complex on which a Lie group  $S$  acts on the left. We call such a space an  $S$ -space. Let  $\pi : ES \rightarrow BS$  be the universal  $S$ -bundle over the classifying space  $BS$ . Then the total space  $ES$  is a contractible space on which  $S$  acts freely on the right. Thus,  $ES \times M$  is a space with the same homotopy type as  $M$ , and  $S$  acts freely on  $ES \times M$  by the diagonal action:

$$(e, x).s = (es, s^{-1}x), \quad \text{for } e \in E, x \in M, s \in S.$$

The quotient space  $(ES \times M)/S$ , denoted  $ES \times_S M$  or  $M_S$ , is called the *homotopy quotient* of  $M$  by  $S$ .

**Definition.** The *equivariant cohomology*  $H_S^*(M)$  of  $M$  under the action of  $S$  is the singular cohomology of the homotopy quotient  $M_S$ .

One may compute equivariant cohomology with any coefficient ring. Since we are interested in manifolds and differential forms, we will take the coefficient ring to be  $\mathbb{R}$ .

The homotopy quotient of a point is the classifying space of  $S$ , since

$$(\text{pt})_S = (ES \times \text{pt})/S = (ES)/S = BS.$$

Hence, the equivariant cohomology of a point is the ordinary cohomology of the classifying space  $BS$ .

A map  $f : M \rightarrow N$  of  $S$ -spaces is  $S$ -equivariant or an  $S$ -map if  $f(s.x) = s.f(x)$  for all  $x$  in  $M$ . An  $S$ -equivariant map  $f : M \rightarrow N$  induces a map  $f_S : M_S \rightarrow N_S$  of homotopy quotients and therefore a homomorphism in equivariant cohomology:

$$f_S^* : H_S^*(N) \rightarrow H_S^*(M).$$

Since every  $S$ -space  $M$  has an  $S$ -equivariant map  $f : M \rightarrow \text{pt}$ , there is a ring homomorphism

$$f_S^* : H_S^*(\text{pt}) \rightarrow H_S^*(M).$$

Thus, the equivariant cohomology ring  $H_S^*(M)$  is a module over  $H^*(BS)$ .

In summary, for a fixed Lie group  $S$ , equivariant cohomology is a contravariant functor from the category of  $S$ -spaces and  $S$ -maps to the category whose objects are rings that are simultaneously  $H^*(BS)$ -modules and whose morphisms are ring homomorphisms that are simultaneously  $H^*(BS)$ -module homomorphisms.

## 2. THE WEIL MODEL

As before, let  $S$  be a Lie group acting on a space  $M$ . When  $M$  is a manifold and the action is  $C^\infty$ , we might ask whether it is possible to compute equivariant cohomology using differential forms, just as the de Rham complex of  $M$  computes its singular cohomology with real coefficients. The Weil model and the Cartan model answer this question affirmatively.

Let  $X_1, \dots, X_\ell$  be a basis for the Lie algebra  $\mathfrak{s}$  of the Lie group  $S$ . The Lie algebra structure of  $\mathfrak{s}$  is completely specified by its structure constants  $c_{ij}^k$ , defined by

$$[X_i, X_j] = \sum_{k=1}^{\ell} c_{ij}^k X_k.$$

Suppose that a Lie group  $S$  acts on a manifold  $P$ . Then each  $Y$  in the Lie algebra  $\mathfrak{s}$  defines a *fundamental vector field*  $\tilde{Y} = \tilde{Y}_P$  on  $P$  as follows: for  $p \in P$ ,

$$\tilde{Y}_p = \begin{cases} \left. \frac{d}{dt} \right|_{t=0} e^{-tY} \cdot p & \text{if } S \text{ acts on } P \text{ on the left;} \\ \left. \frac{d}{dt} \right|_{t=0} p \cdot e^{tY} & \text{if } S \text{ acts on } P \text{ on the right.} \end{cases}$$

If  $S$  acts on the left on  $P$  and  $i_p : S \rightarrow P$  is the map  $i_p(s) = s \cdot p$ , then  $\tilde{Y}_p = (i_p)_*(-Y)$ ; similarly, if  $S$  acts on the right on  $P$  and  $i_p(s) = p \cdot s$ , then  $\tilde{Y}_p = (i_p)_*(Y)$ . Our choice of signs ensures that the construction of fundamental vector fields is a Lie algebra homomorphism, whether the group  $S$  acts on the left or on the right: for  $X, Y \in \mathfrak{s}$ ,

$$(1) \quad [X, Y]^\sim = [\tilde{X}, \tilde{Y}].$$

Recall that a *connection* on a principal  $S$ -bundle  $\pi : P \rightarrow M$  is an  $\mathfrak{s}$ -valued 1-form  $\omega$  on  $P$  such that

- a) if  $\tilde{Y}_P$  is the fundamental vector field on  $P$  associated to  $Y \in \mathfrak{s}$ , then  $\omega(\tilde{Y}_P) = Y$ , and
- b) if  $r_s$  is right translation by  $s \in S$ , then  $r_s^* \omega = (\text{Ad } s^{-1}) \omega$ .

In terms of a basis  $X_1, \dots, X_\ell$  for the Lie algebra  $\mathfrak{s}$ ,

$$\omega = \sum \omega_i X_i,$$

where  $\omega_1, \dots, \omega_\ell$  are real-valued 1-forms on  $P$ . In differential geometry one shows that the connection forms  $\omega_i$  and the curvature forms  $\Omega_i$  satisfy the equations

- 1)  $d\omega_i + \frac{1}{2} \sum_{j,k} c_{jk}^i \omega_j \wedge \omega_k = \Omega_i$ ,
- 2)  $d\Omega_i = \sum_{j,k} c_{jk}^i \Omega_j \wedge \Omega_k$ .

For differential forms  $\omega = \sum \omega_i X_i$  and  $\tau = \sum \tau_j X_j$  with values in the Lie algebra  $\mathfrak{s}$ , we introduce the notation

$$[\omega, \tau] = \sum \omega_i \wedge \tau_j [X_i, X_j].$$

Then we may rewrite 1) and 2) as

- 1')  $d\omega + \frac{1}{2} [\omega, \omega] = \Omega$ ,
- 2')  $d\Omega = [\Omega, \omega]$ .

The *Weil algebra*  $W(\mathfrak{s})$  of the Lie algebra  $\mathfrak{s}$  is defined to be the tensor product of the symmetric algebra and the exterior algebra of  $\mathfrak{s}^*$ :

$$W(\mathfrak{s}) = S^*(\mathfrak{s}^*) \otimes \Lambda^*(\mathfrak{s}^*).$$

Fix a basis  $X_1, \dots, X_\ell$  for  $\mathfrak{s}$ . To distinguish the elements of  $S^*(\mathfrak{s}^*)$  from those in  $\Lambda^*(\mathfrak{s}^*)$ , we will write the dual basis for  $\mathfrak{s}^*$  in  $S^*(\mathfrak{s}^*)$  as  $u_1, \dots, u_\ell$ , and the dual basis for  $\mathfrak{s}^*$  in  $\Lambda^*(\mathfrak{s}^*)$  as  $\theta_1, \dots, \theta_\ell$ . We give the Weil algebra a grading by

$$\deg u_i = 2, \quad \deg \theta_i = 1.$$

Define the *Weil differential* on  $W(\mathfrak{s})$  to be the antiderivation  $d$  such that

$$d\theta_i + \frac{1}{2} \sum_{j,k} c_{jk}^i \theta_j \theta_k = u_i, \quad du_i = \sum_{j,k} c_{jk}^i u_j \theta_k.$$

For each  $X \in \mathfrak{s}$ , define an antiderivation  $\iota_X$  on  $W(\mathfrak{s})$  by

$$\iota_X \theta_i = \theta_i(X), \quad \iota_X u_i = 0.$$

Finally, define the Lie derivative  $\mathcal{L}_X$  on  $W(\mathfrak{s})$  by the formula

$$\mathcal{L}_X = d\iota_X + \iota_X d.$$

Because the de Rham complex  $\Omega^*(M)$  of the  $S$ -manifold  $M$  also has these operators  $d$ ,  $\iota_X$  and  $\mathcal{L}_X$ , the tensor product  $W(\mathfrak{s}) \otimes \Omega^*(M)$  inherits the same operators in the usual way. An element  $\alpha$  of  $W(\mathfrak{s}) \otimes \Omega^*(M)$  is said to be *basic* if

$$\iota_X \alpha = 0 \quad \text{and} \quad \mathcal{L}_X \alpha = 0.$$

The first condition says  $\alpha$  is *horizontal*, and the second says  $\alpha$  is *invariant*. It is easily shown that if  $\alpha$  is basic, then so is  $d\alpha$ . Hence, the space of basic forms,  $\text{Basic}(W(\mathfrak{s}) \otimes \Omega^*(M))$ , is a differential complex under  $d$ .

**Theorem 1.** *Suppose a Lie group  $S$  acts on the manifold  $M$ . Then*

$$H^*\{\text{Basic}(W(\mathfrak{s}) \otimes \Omega^*(M))\} \simeq H_S^*(M).$$

We call the differential complex  $\text{Basic}(W(\mathfrak{s}) \otimes \Omega^*(M))$  the *Weil model* for the equivariant cohomology of  $M$  under the action of  $S$ .

### 3. THE CARTAN MODEL

An element of the Weil model is a finite sum

$$\alpha = a + \sum \theta_i a_i + \sum_{i < j} \theta_i \theta_j a_{ij} + \cdots + \theta_1 \cdots \theta_n a_{1 \dots n},$$

where  $a, a_i, a_{ij}, \dots \in S^*(\mathfrak{s}^*) \otimes \Omega^*(M)$ . We now write  $\iota_i$  for  $\iota_{X_i}$ . The horizontality condition implies that

$$a_i = -\iota_i a, \quad a_{ij} = \iota_i a_j = -\iota_i \iota_j a,$$

and in general,

$$(2) \quad a_{i_1 \dots i_k} = (-1)^k \iota_{i_1} a_{i_2 \dots i_k}.$$

By induction, all the coefficients  $a_I$  of a horizontal form  $\alpha$  are determined by  $a \in S^*(\mathfrak{s}^*) \otimes \Omega^*(M)$ .

The Lie group  $S$  acts on the symmetric algebra  $S^*(\mathfrak{s}^*)$  by the coadjoint representation and on  $\Omega^*(M)$  by pulling back forms. Hence,  $S$  also acts on the tensor product  $S^*(\mathfrak{s}^*) \otimes \Omega^*(M)$ . We denote the invariant subspace by  $(S^*(\mathfrak{s}^*) \otimes \Omega^*(M))^S$ .

It follows from (2) that every element of  $\text{Basic}(W(\mathfrak{s}^*) \otimes \Omega^*(M))$  can be written in the form

$$(3) \quad \alpha = a - \sum \theta_i \iota_i a + \sum_{i < j} (\theta_i \iota_i)(\theta_j \iota_j) a - \dots$$

$$(4) \quad = (1 - \theta_1 \iota_1) \cdots (1 - \theta_n \iota_n) a$$

for a unique  $a \in (S^*(\mathfrak{s}^*) \otimes \Omega^*(M))^S$ .

**Theorem 2.** *There is an isomorphism, called the Mathai-Quillen isomorphism,*

$$\begin{aligned} \text{Basic}(W(\mathfrak{s}^*) \otimes \Omega^*(M)) &\leftrightarrow (S^*(\mathfrak{s}^*) \otimes \Omega^*(M))^S, \\ \alpha &\mapsto a \quad (\text{as in Eq. (3)}) \\ (1 - \theta_1 \iota_1) \cdots (1 - \theta_n \iota_n) a &\leftarrow a. \end{aligned}$$

Under this isomorphism, the Weil differential  $d$  corresponds to the Cartan differential

$$d_C a = da - \sum u_i \iota_i a$$

on  $(S^*(\mathfrak{s}^*) \otimes \Omega^*(M))^S$ .

Thus, the differential complex  $\{(S^*(\mathfrak{s}^*) \otimes \Omega^*(M))^S, d_C\}$  also computes the equivariant cohomology  $H_S^*(M)$ . It is called the *Cartan model* of  $H_S^*(M)$ .

For a circle action on  $M$ , we fix an element  $X_1 = X \in \text{Lie}(S^1)$  and choose  $\theta$  to be its dual 1-form. Since  $S^1$  is abelian, its coadjoint representation on  $S^*(\mathfrak{s}^*)$  is trivial. Hence, the Cartan model is  $\Omega^*(M)^{S^1}[u]$ . The element of the Weil model corresponding to the element  $a$  in the Cartan model is

$$\alpha = a - \theta \iota_X a, \quad a \in \Omega^*(M)^{S^1}[u].$$

#### 4. FUNDAMENTAL VECTOR FIELDS OF THE $S$ -ACTION

Let  $\pi : P \rightarrow M$  be an  $S$ -equivariant principal  $G$ -bundle. By hypothesis,  $S$  acts on the left on  $P$ , and  $G$  acts on the right on  $P$ , and the two actions commute.

**Proposition 3.** *For  $X \in \mathfrak{s}$ , let  $\tilde{X} = \tilde{X}_P$  be the fundamental vector field corresponding to  $X$  under the  $S$ -action. Then  $\tilde{X}$  is invariant under  $G$  and transforms by the adjoint representation under  $S$ ; more precisely,*

- i) for  $g \in G$ ,  $r_{g*}(\tilde{X}) = \tilde{X}$ ;
- ii) for  $s \in S$ ,  $\ell_{s*}(\tilde{X}) = ((\text{Ad } s)X)^\sim$ .

*Proof.* i) Let  $i_p : S \rightarrow P$  be the map  $i_p(s) = s.p$ . Then  $r_g \circ i_p = i_{pg}$ . Hence,

$$r_{g*}\tilde{X}_p = r_{g*}i_{p*}(-X) = i_{pg*}(-X) = \tilde{X}_{pg}.$$

ii) For  $s, t \in S$ ,

$$\ell_s \circ i_p(t) = stp = sts^{-1}sp = i_{sp} \circ c_s(t).$$

where  $c_s(t) = sts^{-1}$  is conjugation by  $s$ . Then

$$\begin{aligned} (\ell_s)_*\tilde{X}_p &= \ell_{s*}i_{p*}(-X) = i_{sp*}c_{s*}(-X) = i_{sp*}(-(\text{Ad } s)X) \\ &= ((\text{Ad } s)X)^\sim_{sp}. \end{aligned}$$

□

If  $\omega$  is a differential form on  $P$  and  $X$  is an element of the Lie algebra  $\mathfrak{s}$ , we define  $\iota_X \omega$  to be the contraction  $\iota_{\tilde{X}} \omega$ , and  $\mathcal{L}_X \omega$  to be the Lie derivative  $\mathcal{L}_{\tilde{X}} \omega$ .

#### 5. THE $\mathfrak{g}$ -VALUED FUNCTION $L_X$

Recall that a *connection* on the principal  $G$ -bundle  $P \rightarrow M$  is a  $\mathfrak{g}$ -valued 1-form  $\Theta$  on  $P$  satisfying the two conditions:

- i)  $\Theta(Z_P^*) = Z$  if  $Z \in \mathfrak{g}$  and  $Z_P^*$  is the fundamental vector field on  $P$  associated to  $Z$  by the  $G$ -action,
- ii)  $r_g^* \Theta = (\text{Ad } g^{-1}) \Theta$  for all  $g \in G$ .

Fix a connection  $\Theta$  on the  $S$ -equivariant principal  $G$ -bundle  $\pi : P \rightarrow M$ . For  $X \in \mathfrak{s}$ , define the function  $L_X : P \rightarrow \mathfrak{g}$  by

$$L_X(p) := -(\iota_X \Theta)(p) := -\Theta_p(\tilde{X}_p).$$

When the principal  $G$ -bundle  $P \rightarrow M$  is  $S$ -equivariant, we say that the connection  $\Theta$  is  $S$ -invariant if the Lie derivative  $\mathcal{L}_X \Theta = 0$  for all  $X \in \mathfrak{s}$ . This is equivalent to  $\ell_s^* \Theta = \Theta$  for all  $s \in S$ . The compactness of  $S$  allows us to average any connection  $\Theta$  over  $S$  to obtain an  $S$ -invariant connection

$$\int_S \ell_s^* \Theta \, ds.$$

Since characteristic classes are independent of the connection, we may assume from the outset, if we like, that the connection  $\Theta$  is  $S$ -invariant.

**Proposition 4.** i) If  $\Theta$  is any connection on  $P$ , then for  $g \in G$ ,  $r_g^* L_X = (\text{Ad } g^{-1}) L_X$ .

ii) If the connection  $\Theta$  is  $S$ -invariant, then for  $s \in \mathfrak{s}$ ,  $\ell_s^* L_X = L_{(\text{Ad } s^{-1})X}$ .

*Proof.* i) By Lemma 3(i), the vector field  $\tilde{X}$  is right-invariant under  $G$ . Thus, for  $p \in P$  and  $g \in G$ ,

$$\begin{aligned} L_X(pg) &= -\Theta_{pg}(\tilde{X}_{pg}) = -\Theta_{pg}(r_{g*} \tilde{X}_p) = -(r_g^* \Theta_{pg})(\tilde{X}_p) \\ &= -(\text{Ad } g^{-1}) \Theta_p(\tilde{X}_p) = (\text{Ad } g^{-1}) L_X(p). \end{aligned}$$

ii)

$$\begin{aligned} (\ell_s^* L_X)(p) &= L_X(sp) = -\Theta_{sp}(\tilde{X}_{sp}) \\ &= -\Theta_{sp}(\ell_{s*}((\text{Ad } s^{-1})X)_p) \quad (\text{by Prop. 3(ii)}) \\ &= -\ell_s^* \Theta_{sp}(((\text{Ad } s^{-1})X)_p) \\ &= -\Theta_p(((\text{Ad } s^{-1})X)_p) \quad (\text{by the } S\text{-invariance of } \Theta) \\ &= L_{(\text{Ad } s^{-1})X}. \end{aligned}$$

□

Although the connection  $\Theta$  is  $S$ -invariant, Prop. 4(ii) shows that the function  $L_X$  is not  $S$ -invariant. Indeed, on the Lie algebra level, we have the following identity.

**Proposition 5.** If  $\Theta$  is an  $S$ -invariant connection on  $P$  and  $X, Y \in \mathfrak{s}$ , then  $\mathcal{L}_Y L_X = L_{[Y, X]}$ .

*Proof.*

$$\begin{aligned} \mathcal{L}_Y L_X &= -\mathcal{L}_Y \Theta(\tilde{X}) \\ &= -(\mathcal{L}_Y \Theta)(\tilde{X}) - \Theta(\mathcal{L}_Y \tilde{X}) \quad (\text{derivation property of the Lie derivative}) \\ &= -\Theta([\tilde{Y}, \tilde{X}]) \quad (\text{by the } S\text{-invariance of } \Theta) \\ &= -\Theta([Y, X]) \quad (\text{by Eq. (1)}) \\ &= L_{[Y, X]}. \end{aligned}$$

□

If  $X_1, \dots, X_\ell$  is a basis for the Lie algebra  $\mathfrak{s}$ , we write  $\iota_i$  and  $L_i$  for  $\iota_{X_i}$  and  $L_{X_i}$ , respectively.

## 6. THE CURVATURE OF A CONNECTION

The curvature of a connection  $\Theta$  on a principal  $G$ -bundle  $P \rightarrow M$  is given by the well-known formula:

$$K = d\Theta + \frac{1}{2}[\Theta, \Theta].$$

In case  $G$  is a matrix group, this is equivalent to

$$K = d\Theta + \Theta \wedge \Theta.$$

By the equivariance property of a connection, it is easily checked that under right translation  $r_g$  by an element  $g$  in  $G$ ,

$$r_g^* K = (\text{Ad } g^{-1})K.$$

In the terminology of Kobayashi and Nomizu ([9]), this shows that  $K$  is a “pseudotensorial 2-form of type  $\text{Ad } G$ ”, and hence

$$K \in \Omega^2(M, \text{Ad } P).$$

In general, we can identify  $\Omega^k(M, \text{Ad } P)$  with the space of horizontal  $\mathfrak{g}$ -valued  $k$ -forms  $\alpha$  on  $P$  satisfying

$$r_g^* \alpha = (\text{Ad } g^{-1})\alpha.$$

Thus, for any connection  $\Theta$  on an  $S$ -equivariant principal bundle  $\pi : P \rightarrow M$  and for any  $X \in \mathfrak{s}$ ,

$$L_X \in \Omega^0(M, \text{Ad } P), \quad \Theta \in \Omega^1(M, \text{Ad } P), \quad K \in \Omega^2(M, \text{Ad } P).$$

A connection on a principal bundle induces a covariant derivative on all associated bundles. In particular, the connection  $\Theta$  on the principal bundle  $P \rightarrow M$  induces a covariant derivative  $D$  on the adjoint bundle:

$$D : \Omega^k(M, \text{Ad } P) \rightarrow \Omega^{k+1}(M, \text{Ad } P), \quad \text{for all } k = 0, 1, 2, \dots$$

It is known that the covariant derivative  $D$  on  $\Omega^k(M, \text{Ad } P)$  is given by the formula

$$D\alpha = d\alpha + [\Theta, \alpha].$$

**Proposition 6.** *Let  $P \rightarrow M$  be an  $S$ -equivariant principal  $G$ -bundle,  $\Theta$  an  $S$ -invariant connection on  $P$ , and  $K$  the curvature form of  $\Theta$ . For  $X, Y \in \mathfrak{s}$ ,*

$$\text{i) } \iota_X K = DL_X = dL_X + [\Theta, L_X].$$

$$\text{ii) } \iota_Y \iota_X K = [L_X, L_Y] - L_{[X, Y]}.$$

*Proof.* i) Applying  $\iota_X$  to both sides of

$$K = d\Theta + \frac{1}{2}[\Theta, \Theta],$$

we get

$$\begin{aligned} \iota_X K &= \iota_X d\Theta + \frac{1}{2}[\iota_X \Theta, \Theta] - \frac{1}{2}[\Theta, \iota_X \Theta] \\ &= \mathcal{L}_X \Theta - d\iota_X \Theta - \frac{1}{2}[L_X, \Theta] + \frac{1}{2}[\Theta, L_X] \quad (\text{because } \mathcal{L}_X = \iota_X d + d\iota_X) \\ &= dL_X + [\Theta, L_X] \quad (\mathcal{L}_X \Theta = 0 \text{ since } \Theta \text{ is } S\text{-invariant}). \end{aligned}$$

This last expression is precisely the covariant derivative  $DL_X$  of  $L_X$  induced by the connection  $\Theta$ , where we view  $D$  as a map from  $\Omega^0(M, \text{Ad } P)$  to  $\Omega^1(M, \text{Ad } P)$ .



ii)

$$\begin{aligned}
\iota_Y \iota_X K &= \iota_Y (dL_X + [\Theta, L_X]) && \text{(by Part (i))} \\
&= (\mathcal{L}_Y - d\iota_Y) L_X + [\iota_Y \Theta, L_X] \\
&= \mathcal{L}_Y L_X - [L_Y, L_X] \\
&= L_{[Y, X]} + [L_X, L_Y] && \text{(Prop. 5)} \\
&= [L_X, L_Y] - L_{[X, Y]}.
\end{aligned}$$

□

## 7. THE CIRCLE CASE

At this point, we specialize to a circle action, not because of any logical necessity, but because when  $S$  is the circle  $S^1$ , the computation is simpler and the ideas of the proof are more transparent. In the circle case, the approximating bundle  $ES(r) \rightarrow BS(r)$  is the Hopf fibration  $S^{2r+1} \rightarrow \mathbb{C}P^r$ .

We view the sphere  $S^{2r+1}$  as the unit sphere in  $\mathbb{C}^{r+1}$ . Then  $S^1$  acts on  $S^{2r+1}$  on the right by scalar multiplication:

$$z.t = (z_0, \dots, z_r).t = (tz_0, \dots, tz_r), \quad t \in S^1, z \in S^{2r+1}.$$

Clearly, the stabilizer at each point  $z \in S^{2r+1}$  is the identity, so this is a free action. Fix a nonzero element  $X$  in  $\mathfrak{s} = \text{Lie}(S^1)$  and let  $\tilde{X} := \tilde{X}_{S^{2r+1}}$  be the fundamental vector field on  $S^{2r+1}$  associated to  $X$ . Let  $\theta$  be a connection for the  $S^1$ -bundle  $S^{2r+1} \rightarrow \mathbb{C}P^r$ . Since  $S^1$  is abelian,

$$r_s^* \theta = (\text{Ad } s^{-1}) \theta = \theta.$$

So the connection  $\theta$  is  $S$ -invariant. In particular,

$$(5) \quad \mathcal{L}_X \theta = 0.$$

Moreover, since fixing  $X$  amounts to choosing an isomorphism  $\text{Lie}(S^1) \simeq \mathbb{R}$  under which  $X$  corresponds to 1,

$$\iota_X \theta = \theta(\tilde{X}) \equiv 1.$$

**7.1. Induced Connection and Curvature on  $P^r \rightarrow M^r$ .** Let  $P \rightarrow M$  be an  $S^1$ -equivariant principal  $G$ -bundle with an  $S^1$ -invariant connection  $\Theta$ . In the circle case, we fix an element  $X \in \text{Lie}(S^1)$  and abbreviate  $L_X := -\iota_X \Theta$  as  $L$ .

We approximate the homotopy quotients  $P_{S^1}$  and  $M_{S^1}$  by

$$P^r := S^{2r+1} \times_{S^1} P, \quad M^r := S^{2r+1} \times_{S^1} M.$$

Then the principal  $G$ -bundle  $P^r \rightarrow M^r$  is a finite-dimensional approximation to  $P_{S^1} \rightarrow M_{S^1}$ . Instead of finding directly a connection on  $P^r \rightarrow M^r$ , we will construct a connection on  $S^{2r+1} \times P \rightarrow S^{2r+1} \times M$  that is basic with respect to the  $S^1$ -action. It will then descend to a connection on  $P^r \rightarrow M^r$ .

**Proposition 7.** *The  $\mathfrak{g}$ -valued 1-form*

$$\Xi = 1 \otimes \Theta + \theta \otimes L$$

*on  $S^{2r+1} \times P$  is a connection form for the principal  $G$ -bundle  $S^{2r+1} \times P \rightarrow S^{2r+1} \times M$  and is basic with respect to the circle action on  $S^{2r+1} \times P$ .*

*Proof.* Since  $L$  and  $\Theta$  are  $\mathfrak{g}$ -valued and  $\theta$  and 1 are scalar-valued,  $\Xi$  is a  $\mathfrak{g}$ -valued 1-form on  $S^{2r+1} \times P$ .

- i) If  $Z \in \mathfrak{g}$  and  $Z_P^*$  is the fundamental vector field on  $P$  associated to  $Z$  under the  $G$ -action, then  $(0, Z_P^*)$  is the fundamental vector field on  $S^{2r+1} \times P$  associated to  $Z$ . Hence,

$$\Xi(0, Z_P^*) = (1 \otimes \Theta + \theta \otimes L)(0, Z_P^*) = \Theta(Z_P^*) = Z.$$

- ii) Right-equivariance: For  $g \in G$ ,

$$\begin{aligned} r_g^* \Xi &= 1 \otimes r_g^* \Theta + (r_g^* \theta) \otimes r_g^* L \\ &= 1 \otimes (\text{Ad } g^{-1}) \Theta + \theta \otimes (\text{Ad } g^{-1}) L \\ &= (\text{Ad } g^{-1})(1 \otimes \Theta + \theta \otimes L) = (\text{Ad } g^{-1}) \Xi. \end{aligned}$$

Hence,  $\Xi$  is a connection on  $S^{2r+1} \times P \rightarrow S^{2r+1} \times M$ .

To prove that  $\Xi$  is basic with respect to the  $S^1$ -action, we check the horizontality condition  $\iota_X \Xi = 0$  and the invariance condition  $\mathcal{L}_X \Xi = 0$ .

- iii) Horizontality:

$$\begin{aligned} \iota_X \Xi &= \iota_X (1 \otimes \Theta + \theta \otimes L) \\ &= 1 \otimes \iota_X \Theta + (\iota_X \theta) \otimes L - \theta \otimes \iota_X L \\ &= -1 \otimes L + 1 \otimes L = 0. \end{aligned}$$

- iv) Invariance:

$$\begin{aligned} \mathcal{L}_X \Xi &= \mathcal{L}_X (1 \otimes \Theta + \theta \otimes L) \\ &= 1 \otimes \mathcal{L}_X \Theta + (\mathcal{L}_X \theta) \otimes L + \theta \otimes \mathcal{L}_X L. \end{aligned}$$

In this sum,  $\mathcal{L}_X \theta = 0$  and  $\mathcal{L}_X \Theta = 0$  because both connections  $\theta$  and  $\Theta$  are  $S^1$ -invariant (Eq. (5)); moreover, by Prop. 5,

$$\mathcal{L}_X L = \mathcal{L}_X L_X = L_{[X, X]} = 0.$$

Hence,  $\mathcal{L}_X \Xi = 0$ . □

So  $\Xi = 1 \otimes \Theta + \theta \otimes L$  descends to  $P^r = S^{2r+1} \times_{S^1} P$ , i.e., if  $\alpha : S^{2r+1} \times P \rightarrow P^r$  is the projection, then  $\Xi = \alpha^* \Xi'$  for some  $\mathfrak{g}$ -valued 1-form  $\Xi'$  on  $P^r$ . From the following lemma, it follows that  $\Xi'$  is a connection for  $\tau : P^r \rightarrow M^r$ .

**Lemma 8.** *Let  $G$  be a Lie group with Lie algebra  $\mathfrak{g}$  and  $\pi : P \rightarrow M$ ,  $\pi' : P' \rightarrow M'$  two principal  $G$ -bundles. Suppose there is a  $G$ -equivariant submersion  $\alpha : P \rightarrow P'$  and  $\Theta'$  is a  $\mathfrak{g}$ -valued 1-form on  $P'$ . Then  $\alpha^* \Theta'$  is a connection for  $\pi : P \rightarrow M$  if and only if  $\Theta'$  is a connection for  $\pi' : P' \rightarrow M'$ .*

The proof of this lemma is based on a series of simple remarks.

**Remark.**

- i) For  $p \in P$ , let  $i_p : G \rightarrow P$  be the map  $i_p(g) = pg$ . Similarly, for  $p' \in P'$ , let  $i_{p'} : G \rightarrow P'$  be the map  $i_{p'}(g) = p'g$ . Then

$$\alpha \circ i_p = i_{\alpha(p)}.$$

- ii) For  $g \in G$ , let  $r_g$  be right translation by  $g$  on a principal  $G$ -bundle. Since  $\alpha : P \rightarrow P'$  is  $G$ -equivariant,

$$r_g \circ \alpha = \alpha \circ r_g.$$

- iii) For  $Y \in \mathfrak{g}$ , if  $Y_P$  and  $Y_{P'}$  are the fundamental vector fields associated to  $Y$  on  $P$  and  $P'$  respectively, then  $Y_{P'} = \alpha_*(Y_P)$ .

*Proof of iii).* For  $p \in P$ ,

$$\alpha_*(Y_P)_p = \alpha_* i_{p*}(Y) = (\alpha \circ i_p)_* Y = i_{\alpha(p)*} Y = (Y_{P'})_{\alpha(p)}.$$

*Proof of Lemma 8.* ( $\Leftarrow$ ) Suppose  $\Theta'$  is a connection for  $\pi' : P' \rightarrow M'$ , and  $Y \in \mathfrak{g}$ . Then

- a)  $(\alpha^* \Theta')(Y_P) = \Theta'(\alpha_* Y_P) = \Theta'(Y_{P'}) = Y.$
- b)  $r_g^* \alpha^* \Theta' = \alpha^* r_g^* \Theta' = \alpha^* (\text{Ad } g^{-1}) \Theta' = (\text{Ad } g^{-1}) \alpha^* \Theta'.$

Hence,  $\alpha^* \Theta'$  is a connection for  $\pi : P \rightarrow M$ .

( $\Rightarrow$ ) Suppose  $\alpha^* \Theta'$  is a connection for  $\pi : P \rightarrow M$ , and  $Y \in \mathfrak{g}$ . Then

- i)  $\Theta'(Y_{P'}) = \Theta'(\alpha_* Y_P) = (\alpha^* \Theta')(Y_P) = Y.$
- ii)  $\alpha^* r_g^* \Theta' = r_g^* \alpha^* \Theta' = (\text{Ad } g^{-1}) \alpha^* \Theta' = \alpha^* ((\text{Ad } g^{-1}) \Theta').$

Since  $\alpha$  is a submersion,  $\alpha^*$  is injective; hence,

$$r_g^* \Theta' = (\text{Ad } g^{-1}) \Theta'.$$

So  $\Theta'$  is a connection for  $\pi' : P' \rightarrow M'$ .

Returning to the situation preceding the lemma, we will identify  $\Xi'$  with  $\Xi$ . This is possible because of the correspondence between forms on  $P^r$  and basic forms on  $S^{2r+1} \times P$ .

**Proposition 9.** *Let  $K$  be the curvature of the connection  $\Theta$  on the  $S^1$ -equivariant principal  $G$ -bundle  $P \rightarrow M$ . Then the curvature  $K_\Xi$  of the induced connection  $\Xi$  on  $P^r \rightarrow M^r$  is the following  $S^1$ -basic form on  $S^{2r+1} \times P$ :*

$$K_\Xi = 1 \otimes K + d\theta \otimes L - \theta \otimes \iota_X K.$$

*Proof.*

$$\begin{aligned} d\Xi &= 1 \otimes d\Theta + (d\theta) \otimes L - \theta \otimes dL, \\ [\Xi, \Xi] &= [1 \otimes \Theta + \theta \otimes L, 1 \otimes \Theta + \theta \otimes L] \\ &= 1 \otimes [\Theta, \Theta] + \theta \otimes [L, \Theta] - \theta \otimes [\Theta, L] \\ &= 1 \otimes [\Theta, \Theta] - \theta \otimes 2[\Theta, L]. \end{aligned}$$

To compute a bracket such as  $[1 \otimes \Theta, \theta \otimes L]$ , first choose a basis  $e_1, \dots, e_n$  for  $\mathfrak{g}$  and write

$$\Theta = \sum \Theta^i e_i, \quad L = \sum L^j e_j,$$

with  $\Theta^i, L^j$  being ordinary forms on  $P$ . Then

$$\begin{aligned} [1 \otimes \Theta, \theta \otimes L] &= \sum [1 \otimes \Theta^i e_i, \theta \otimes L^j e_j] \\ &= \sum \Theta^i \theta L^j [e_i, e_j] \\ &= - \sum \theta \Theta^i L^j [e_i, e_j] \\ &= -\theta [\Theta^i e_i, L^j e_j] \\ &= -\theta [\Theta, L] \\ &= -\theta \otimes [\Theta, L]. \end{aligned}$$

Therefore,

$$\begin{aligned}
K_\Xi &= d\Xi + \frac{1}{2}[\Xi, \Xi] \\
&= 1 \otimes (d\Theta + \frac{1}{2}[\Theta, \Theta]) + (d\theta) \otimes L - \theta \otimes (dL + [\Theta, L]) \\
&= 1 \otimes K + d\theta \otimes L - \theta \otimes DL \\
&= 1 \otimes K + d\theta \otimes L - \theta \otimes \iota_X K \quad (\text{by Lemma 6}).
\end{aligned}$$

Since  $\Xi$  is basic with respect to the  $S^1$ -action, it follows easily that  $\iota_X K_\Xi = 0$  and  $\mathcal{L}_X K_\Xi = 0$ . Hence,  $K_\Xi$  is also basic with respect to the  $S^1$ -action.  $\square$

**7.2. Equivariant Curvature and Equivariant Characteristic Classes.** In Proposition 9, the forms  $\Xi$  and  $\theta$  depend on  $r$ , but  $K$  and  $L$  do not. To indicate this dependence, we write  $K_r$  for  $K_\Xi$  and  $\theta_r$  for  $\theta$ . Let  $u_r = d\theta_r$ . Then, dropping the tensor product sign, the curvature of the induced connection on  $P^r \rightarrow M^r$  is

$$K_r = K + u_r L - \theta_r(\iota_X(K + u_r L)) \in \Omega^*(P^r).$$

Under the sequence of inclusions

$$\begin{array}{ccccccc}
P^1 & \rightarrow & P^2 & \rightarrow & \cdots & P^r & \rightarrow & \cdots & \rightarrow & P_{S^1} \\
\downarrow & & \downarrow & & & \downarrow & & & & \downarrow \\
M^1 & \rightarrow & M^2 & \rightarrow & \cdots & M^r & \rightarrow & \cdots & \rightarrow & M_{S^1}
\end{array},$$

an element in the Weil model of  $P_{S^1}$  that restricts to all  $K_r$  is

$$K_\infty := K + uL - \theta(\iota_X(K + uL)),$$

where  $u$  and  $\theta$  now have their usual meaning in the Weil model. Moreover, by ([8], Prop. 3.52), in any dimension  $k$  the real cohomology of the homotopy quotient  $M_{S^1}$  is the inverse limit of  $H^k(M^r)$  as  $r \rightarrow \infty$ .

**Proposition 10.** *If  $f$  is an  $\text{Ad}(G)$ -invariant polynomial on  $\mathfrak{g}$ , then  $f(K_\infty) \in W(\mathfrak{s}) \otimes \Omega^*(P)$  is basic with respect to both the  $S^1$ -action and the  $G$ -action on  $P$ .*

*Proof.* To check that  $K_\infty$  is basic with respect to the  $S^1$ -action, we compute for  $X \in \text{Lie}(S^1)$ :

$$\begin{aligned}
\iota_X K_\infty &= \iota_X K + u\iota_X L - \iota_X(K + uL) = \iota_X K - \iota_X K = 0, \\
\mathcal{L}_X K_\infty &= \mathcal{L}_X K + u\mathcal{L}_X L - \theta\mathcal{L}_X \iota_X(K + uL) \\
&\quad (\text{in the circle case, } \mathcal{L}_X u = \mathcal{L}_X \theta = 0) \\
&= 0 \quad (\text{since } \mathcal{L}_X K = 0, \mathcal{L}_X L = 0, \text{ and } \mathcal{L}_X \iota_X = \iota_X \mathcal{L}_X).
\end{aligned}$$

To check that  $K_\infty$  is basic with respect to the  $G$ -action, let  $Z \in \mathfrak{g}$ . Since  $K$  and  $L$  are both horizontal with respect to  $G$ ,  $\iota_Z K_\infty = 0$ .

If  $g \in G$ , then by the  $G$ -equivariance of  $K$  and  $L$ ,

$$r_g^* K_\infty = (\text{Ad } g^{-1}) K_\infty.$$

Hence,

$$r_g^* f(K_\infty) = f(r_g^* K_\infty) = f((\text{Ad } g^{-1}) K_\infty) = f(K_\infty).$$

Since  $f(K_\infty)$  is  $G$ -invariant,  $L_Z K_\infty = 0$  for any  $Z \in \mathfrak{g}$ .  $\square$

It follows from this proposition that  $f(K_\infty)$  is an element of  $\text{Basic}_{S^1}(W(\mathfrak{s}) \otimes \Omega^*(P))$  that descends to  $\text{Basic}_{S^1}(W(\mathfrak{s}) \otimes \Omega^*(M))$ . As it is the unique element in the Weil model that restricts to  $f(K_r)$  for all  $r$ , its cohomology class represents the equivariant characteristic class  $c_f(P)$ .

Recall that the recipe for going from the Cartan model to the Weil model of the equivariant cohomology of a circle action is

$$a \mapsto a - \theta(\iota_X a).$$

This shows that the equivariant characteristic class of  $P \rightarrow M$  corresponding to an  $\text{Ad}(G)$ -invariant polynomial  $f$  on  $\mathfrak{g}$  is represented by  $f(K + uL)$  in the Cartan model of  $H_{S^1}^*(M)$ . In particular,  $K + uL$  is the correct notion of *equivariant curvature* in the Cartan model.

## 8. THE GENERAL CASE

Returning to the general case of a compact connected Lie group  $S$ , we approximate the universal  $S$ -bundle  $ES \rightarrow BS$  by principal  $S$ -bundles of finite-dimensional manifolds

$$\begin{array}{ccccccc} ES(1) & \hookrightarrow \dots \hookrightarrow & ES(r) & \hookrightarrow & ES(r+1) & \hookrightarrow \dots \hookrightarrow & ES \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ BS(1) & \hookrightarrow \dots \hookrightarrow & BS(r) & \hookrightarrow & BS(r+1) & \hookrightarrow \dots \hookrightarrow & BS \end{array}$$

On each principal  $S$ -bundle  $ES(r) \rightarrow BS(r)$ , choose a connection  $\theta(r)$ . Now fix an  $r$  and for simplicity write  $\theta$  instead of  $\theta(r)$ .

Let  $\tilde{X}_1, \dots, \tilde{X}_\ell$  denote the fundamental vector fields on  $ES(r)$  corresponding to the basis  $X_1, \dots, X_\ell$  for the Lie algebra  $\mathfrak{s}$ . If  $\theta = \sum \theta_i X_i$ , then

- i)  $\theta(\tilde{X}_j) = \sum \theta_i(\tilde{X}_j)X_i = X_j \Rightarrow \theta_i(\tilde{X}_j) = \delta_{ij}$ .
- ii) for  $s \in S$ ,  $r_s^* \theta = (\text{Ad } s^{-1})\theta$ .

As before, we adopt the shorthand  $\iota_i = \iota_{X_i}$ ,  $\mathcal{L}_i = \mathcal{L}_{X_i}$ , and  $L_i = L_{X_i}$ .

**Proposition 11.** *Let  $c_{ij}^k$  be the structure constants of the Lie algebra  $\mathfrak{s}$  with respect to the basis  $X_1, \dots, X_\ell$ :  $[X_i, X_j] = \sum c_{ij}^k X_k$ . If  $\theta = \sum \theta_i X_i$  is a connection on a principal  $S$ -bundle, then*

$$\mathcal{L}_i \theta_j = - \sum c_{ik}^j \theta_k.$$

*Proof.* Since  $\theta_j(\tilde{X}_k) = \delta_{jk}$ ,

$$\begin{aligned} 0 &= \mathcal{L}_i(\theta_j(\tilde{X}_k)) \\ &= (\mathcal{L}_i \theta_j)(\tilde{X}_k) + \theta_j(\mathcal{L}_i \tilde{X}_k). \end{aligned}$$

Hence,

$$(\mathcal{L}_i \theta_j) \tilde{X}_k = -\theta_j([\tilde{X}_i, \tilde{X}_k]) = -\theta_j(\sum c_{ik}^\alpha \tilde{X}_\alpha) = -c_{ik}^j.$$

If  $\mathcal{L}_i \theta_j = \sum b_{ik}^j \theta_k$ , then  $b_{ik}^j = (\mathcal{L}_i \theta_j)(\tilde{X}_k) = -c_{ik}^j$ . □

Let  $\Theta$  be an  $S$ -invariant connection on the  $S$ -equivariant principal  $G$ -bundle  $\pi : P \rightarrow M$ . Define on  $ES(r) \times P$  the  $\mathfrak{g}$ -valued 1-form

$$\Xi = 1 \otimes \Theta + \sum \theta_i \otimes L_i.$$

**Theorem 12.** *The  $\mathfrak{g}$ -valued 1-form  $\Xi$  is a connection for the principal  $G$ -bundle*

$$ES(r) \times P \rightarrow ES(r) \times M.$$

*Moreover,  $\Xi$  is basic with respect to the  $S$ -action on  $ES(r) \times P$ .*

*Proof.* First we check that  $\Xi$  is a connection on  $ES(r) \times P$ .

- i) Let  $Z \in \mathfrak{g}$  and let  $Z_P^*$  be its fundamental vector field on  $P$ . Then the fundamental vector field of  $Z$  on  $ES(r) \times P$  is  $(0, Z_P^*)$ , because  $G$  acts trivially on the first factor. Hence,

$$\Xi(0, Z_P^*) = 1 \cdot \Theta(Z_P^*) + 0 = Z.$$

- ii) Right-equivariance:

$$\begin{aligned} r_g^* \Xi &= 1 \otimes r_g^* \Theta + \sum \theta_i \otimes r_g^* L_i = (\text{Ad } g^{-1})(1 \otimes \Theta + \sum \theta_i \otimes L_i) \\ &= (\text{Ad } g^{-1}) \Xi. \end{aligned}$$

Next we check that  $\Xi$  is basic with respect to the  $S$ -action.

- iii) Horizontality:

$$\begin{aligned} \iota_{X_j} \Xi &= 1 \otimes \iota_{X_j} \Theta + \sum (\iota_{X_j} \theta_i) \otimes L_i + \sum \theta_i \otimes \iota_{X_j} L_i \\ &= -1 \otimes L_j + 1 \otimes L_j + 0 = 0. \end{aligned}$$

- iv)  $S$ -invariance:

$$\begin{aligned} \mathcal{L}_{X_j} \Xi &= 1 \otimes \mathcal{L}_{X_j} \Theta + \sum (\mathcal{L}_{X_j} \theta_i) \otimes L_i + \sum \theta_i \otimes \mathcal{L}_{X_j} L_i \\ &= 0 + \sum -c_{jk}^i \theta_k \otimes L_i + \sum \theta_i \otimes L_{[X_j, X_i]} \quad (\text{by Prop. 5}) \\ &= -\sum c_{jk}^i \theta_k \otimes L_i + \sum c_{ji}^k \theta_i \otimes L_k = 0. \end{aligned}$$

□

Let  $P^r = ES(r) \times_S P$  and  $M^r = ES(r) \times_S M$ . The principal  $G$ -bundle  $P^r \rightarrow M^r$  is our finite-dimensional approximation to the principal  $G$ -bundle of homotopy quotients  $P_S \rightarrow M_S$ . By Lemma 8,  $\Xi$  is the pullback of a connection  $\Xi'$  on  $P^r \rightarrow M^r$ . We will identify  $\Xi$  with  $\Xi'$ .

**Proposition 13.** *Let  $K$  be the curvature of the  $S$ -invariant connection  $\Theta$  on the  $S$ -equivariant principal  $G$ -bundle  $P \rightarrow M$ . Then the curvature  $K_\Xi$  of the induced connection  $\Xi$  on  $P^r \rightarrow M^r$  is given by the following  $S$ -basic form on  $ES(r) \times P$ :*

$$(6) \quad K_\Xi = 1 \otimes K + \sum (d\theta_i) \otimes L_i - \sum \theta_i \otimes \iota_i K + \sum_{i < j} \theta_i \theta_j \otimes [L_i, L_j].$$

*Proof.*

$$\begin{aligned} d\Xi &= 1 \otimes d\Theta + \sum (d\theta_i) \otimes L_i - \sum \theta_i \otimes dL_i. \\ [\Xi, \Xi] &= [1 \otimes \Theta + \sum \theta_i \otimes L_i, 1 \otimes \Theta + \sum \theta_i \otimes L_i] \\ &= 1 \otimes [\Theta, \Theta] + \sum \theta_i \otimes [L_i, \Theta] - \sum \theta_i \otimes [\Theta, L_i] + \sum_{i,j} \theta_i \theta_j \otimes [L_i, L_j] \\ &= 1 \otimes [\Theta, \Theta] - \sum \theta_i \otimes 2[\Theta, L_i] + \sum_{i,j} \theta_i \theta_j \otimes [L_i, L_j] \end{aligned}$$

In the last term, if we sum over only  $i, j$  such that  $i < j$ , instead of all  $i, j = 1, \dots, \ell$ , it becomes

$$\sum_{i,j=1}^{\ell} \theta_i \theta_j \otimes [L_i, L_j] = 2 \sum_{i < j} \theta_i \theta_j \otimes [L_i, L_j].$$

Hence,

$$\begin{aligned} K_{\Xi} &= d\Xi + \frac{1}{2}[\Xi, \Xi] \\ &= 1 \otimes K + \sum (d\theta_i) \otimes L_i - \sum \theta_i \otimes (dL_i + [\Theta, L_i]) + \sum_{i < j} \theta_i \theta_j \otimes [L_i, L_j]. \end{aligned}$$

The final formula now follows from Prop. 6(i).  $\square$

For simplicity, we drop the tensor product symbol in Eq. (6). Let

$$u_k = d\theta_k + \sum_{i < j} c_{ij}^k \theta_i \theta_j$$

be the curvature of the connection  $\theta$  on  $ES(r) \rightarrow BS(r)$ . We can rewrite (6) as

$$\begin{aligned} K_{\Xi} &= K + \sum_k (d\theta_k + \sum_{i < j} c_{ij}^k \theta_i \theta_j) L_k - \sum \theta_i \iota_i K \\ &\quad + \sum_{i < j} \theta_i \theta_j [L_i, L_j] - \sum_k \sum_{i < j} \theta_i \theta_j c_{ij}^k L_k \\ (7) \quad &= K + \sum u_k L_k - \sum \theta_i \iota_i K + \sum_{i < j} \theta_i \theta_j [L_i, L_j] - \sum_i \sum_{i < j} \theta_i \theta_j c_{ij}^k L_k. \end{aligned}$$

The last two terms of (7) can be simplified somewhat, for

$$\sum_k c_{ij}^k L_k = \sum_k c_{ij}^k L_{X_k} = L_{\sum c_{ij}^k X_k} = L_{[X_i, X_j]};$$

hence,

$$\begin{aligned} [L_i, L_j] - \sum_k c_{ij}^k L_k &= [L_{X_i}, L_{X_j}] - L_{[X_i, X_j]} \\ &= \iota_{X_j} \iota_{X_i} K \quad (\text{by Prop. 6(ii)}). \end{aligned}$$

Because  $\iota_i$  anticommutes with  $\iota_j$  and  $\theta_j$ ,  $\theta_i \theta_j \iota_j \iota_i = \theta_i \iota_i \theta_j \iota_j$  and

$$K_{\Xi} = K + \sum u_k L_k - \sum \theta_i \iota_i K + \sum_{i < j} \theta_i \iota_i \theta_j \iota_j K.$$

Since  $L_k$  is a 0-form,  $\iota_i L_k = 0$  for all  $i$ . Similarly, since  $K$  is a 2-form, contracting  $K$  three or more times with vector fields yields 0, for example,  $\iota_{i_1} \iota_{i_2} \iota_{i_3} K = 0$ . Hence,

$$\begin{aligned} K_{\Xi} &= (K + \sum u_k L_k) - \sum \theta_i \iota_i (K + \sum u_k L_k) \\ &\quad + \sum \theta_i \iota_i \theta_j \iota_j (K + \sum u_k L_k) - \dots \\ (8) \quad &= \prod_{i=1}^{\ell} (1 - \theta_i \iota_i) (K + \sum u_k L_k). \end{aligned}$$

As in the circle case, in this formula  $K_{\Xi}$ ,  $\theta_i$  and  $u_k$  are all differential forms on  $P^r$  and should be more properly written as  $K_{\Xi}(r)$ ,  $\theta_i(r)$  and  $u_k(r)$  to indicate their dependence on  $r$ . If  $f$  is an  $\text{Ad } G$ -invariant polynomial on  $\mathfrak{g}$ , then  $f(K_{\Xi}(r))$

represents the characteristic class  $c_f(P^r)$ . Clearly, the element in the Weil model of  $P_S$  that restricts to all  $f(K_{\Xi}(r))$  is  $f(K_{\infty})$ , where  $K_{\infty}$  is given by the same formula as (8),

$$K_{\infty} = \prod_{i=1}^{\ell} (1 - \theta_i \iota_i) (K + \sum u_k L_k),$$

but  $\theta_i$  and  $u_k$  now have their usual meaning as elements of the Weil algebra of  $S$ .

The rest of the argument proceeds as in the circle case. The upshot is that  $f(K_{\infty})$  is the element in the Weil model of  $H_S^*(M)$  which restricts to the characteristic classes  $c_f(P^r)$  for all  $r$ , and therefore represents the equivariant characteristic class  $c_f(P)$ . Under the Mathai-Quillen isomorphism, the corresponding element in the Cartan model is  $f(K + \sum u_k L_k)$ .

An element of the Cartan model  $(S^*(\mathfrak{s}^*) \otimes \Omega^*(M))^S$  may be viewed as a polynomial function  $h : \mathfrak{s} \rightarrow \Omega^*(M)$  which is  $S$ -equivariant in the following sense:

$$h((\text{Ad } s)X) = (s^{-1})^* h(X).$$

From this point of view,  $f(K + \sum u_k L_k)$  is the function:  $\mathfrak{s} \rightarrow \Omega^*(M)$  given by

$$(9) \quad X \mapsto f(K + L_X) \in \Omega^*(M),$$

since

$$L_X = L_{\sum u_k(X) X_k} = \sum u_k(X) L_{X_k} = \sum u_k L_k(X).$$

By Prop. 4,

$$L_{(\text{Ad } s)X} = (s^{-1})^* L_X,$$

which shows that  $K + L_X$  is  $S$ -invariant and therefore  $f(K + L_X)$  is an  $S$ -invariant element of  $S^*(\mathfrak{s}^*) \otimes \Omega^*(M)$ . Finally, if we denote the function in (9) by  $K + L$ , then the element of the Cartan model of  $M_S$  corresponding to the equivariant characteristic class  $c_f(P_S)$  is  $f(K + L)$ .

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